A Prototype Finite Difference Model

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A Prototype Model

We will introduce a finite difference model that will serve to demonstrate what a computational scientist needs to do to take advantage of Distributed Memory computers using MPI

The model we are using is a two dimensional solution to a model problem for Ocean Circulation

The Prototype Model: The Stommel Problem

Wind-driven circulation in a homogeneous rectangular ocean under the influence of surface winds, linearized bottom friction, flat bottom and Coriolis force.

Solution: intense crowding of streamlines towards the western boundary caused by the variation of the Coriolis parameter with latitude

Governing Equations and Model Constants

$\psi = 0$

$$\gamma\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right) + \beta \frac{\partial \psi}{\partial x} = f$$

$$f = -\alpha \sin\left(\frac{\pi y}{L_y}\right)$$

$\psi = 0$

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$$L_x = L_y = 2000 \ Km$$

$$\gamma = 3 \times 10^{(-6)}$$

$$\beta = 2.25 \times 10^{(-11)}$$

$$\alpha = 10^{(-9)}$$

$\psi = 0$

Domain Discretization

Define a grid consisting of points (x_i, y_j) given by

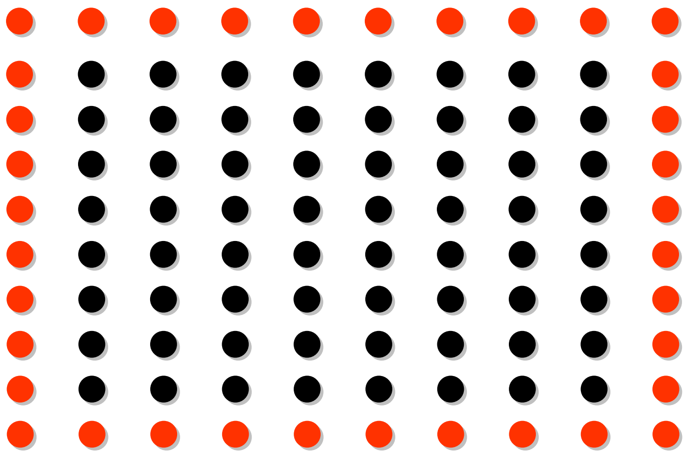
$$x_i = i\Delta x, i = 1,2,\ldots\ldots., nx$$

$$y_j = j\Delta y, \; j = 1,2,\ldots\ldots., ny$$

$$\Delta x = L_x \big/ (nx - 1)$$

$$\Delta y = L_y \big/ (ny - 1)$$

Domain Discretization



seek to find an approximation to $\psi(x_i, \mathrm{y}_{\mathrm{j}})$ at points (x_i, y_j):

$$\psi_{i,j} \approx \psi(x_i, y_j)$$

Centered Finite Difference Scheme for the Derivative Operators

$$\frac{\partial \psi}{\partial x} \approx \frac{\psi_{i+1,j} - \psi_{i-1,j}}{2\Delta x}$$

$$\frac{\partial^2 \psi}{\partial x^2} \approx \frac{\psi_{i+1} - 2\psi_{i,j} + \psi_{i-1,j}}{(\Delta x)^2}$$

$$\frac{\partial^2 \psi}{\partial y^2} \approx \frac{\psi_{i,j+1} - 2\psi_{i,j} + \psi_{i,j-1}}{(\Delta y)^2}$$

Finite Difference Form of the Governing Equation

$$\psi_{i,j} = a_1\psi_{i+1,j} + a_2\psi_{i-1,j} + a_3\psi_{i,j+1} + a_4\psi_{i,j-1} - a_5 f_{i,j}$$

$$a_1 = \frac{\Delta y^2}{2(\Delta x^2 + \Delta y^2)} + \frac{\beta \Delta x^2 \Delta y^2}{4\gamma \Delta x(\Delta x^2 + \Delta y^2)}$$

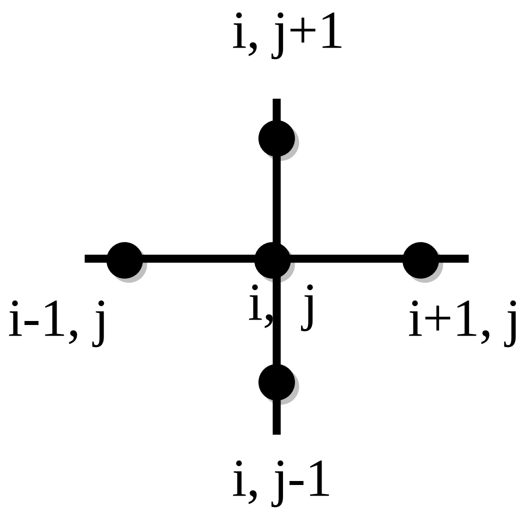
$$a_2 = \frac{\Delta y^2}{2(\Delta x^2 + \Delta y^2)} - \frac{\beta \Delta x^2 \Delta y^y}{4\gamma \Delta x(\Delta x^2 + \Delta y^2)}$$

$$a_3 = \frac{\Delta x^2}{2(\Delta x^2 + \Delta y^2)}$$

$$a_4 = \frac{\Delta x^2}{2(\Delta x^2 + \Delta y^2)}$$

$$a_5 = \frac{\Delta x^2 \Delta y^2}{2\gamma(\Delta x^2 + \Delta y^2)}$$

Five-point Stencil Approximation for the Discrete Stommel Model



interior grid points:
i=2,nx-1 ; j=2,ny-1

boundary points:

(i,1) & (i,ny) ; i=1,nx

(1,j) & (nx,j) ; j=1,ny

$$\psi_{i,j} = a_1\psi_{i+1,j} + a_2\psi_{i-1,j} + a_3\psi_{i,j+1} + a_4\psi_{i,j-1} - a_5 f_{i,j}$$

$$\psi_{i,1} = \psi_{i,ny} = 0; \;::\; \psi_{1,j} = \psi_{nx,j} = 0$$

Jacobi Iteration

Start with an initial guess for $(\psi_{i,j})$

do i = 2, nx-1 ; j = 2, ny-1

$$(\psi_{i,j})\_new = a_1(\psi_{i+1,j}) + a_2(\psi_{i-1,j}) + a_3(\psi_{i,j+1}) + a_4(\psi_{i,j-1}) - a_5 f_{i,j}$$

end do

Copy $(\psi_{i,j}) = (\psi_{i,j})\_new$

Repeat the process